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Arithmetic and geometric solutions for average rigid-body rotation ☆

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ABSTRACT

Several existing formulations for the rotation average are reviewed and classified into the Euclidean and Riemannian solutions. A novel, more efficient characterization of the Riemannian-based average is proposed. The discussion addresses the issue of bi-invariance of the underlying distance metrics, and how the different solutions are interrelated. A not bi-invariant arithmetic average of rotation vectors is considered and shown to be an approximate solution to both the Riemannian and Euclidean averages. Results for four numerical examples are presented demonstrating the closeness of all solutions in practical applications, but also their differences when the rotations to be averaged are orthogonal to each other.

1. Introduction

1.1. Background and motivation

The need to calculate an average of several rigid-body rotations arises in a number of applications. In robotics, for example, the ubiquitous use of cameras and their low cost make it practical to equip robotic systems with multiple cameras. These may be used to determine the pose of objects in the environment or the pose of robot end-effector and provide multiple measurements of the same. Another use of rotation averaging and more generally orientation *statistics*, is illustrated in Ref. [1], the authors of which apply their statistical approach to analyze human upper limb poses in a drilling task.

Our motivation for investigating the present subject arose from the research in human gait analysis. In this context, researchers usually collect measurements with a motion capture (MOCAP) system which generates 3D positions of markers mounted on the subject's body [2–4]. The data is then post-processed with essentially an inverse kinematics algorithm to reconstruct the joint kinematics of the body from the measured marker coordinates. One complicating factor in this procedure is the soft tissue artifact: it corrupts the validity of the rigid-body approximation to the motion of the markers. A number of algorithms have been proposed to deal with this specific problem [5–7]. One possible approach is to use *patches* of markers, the motions of which individually best match that of a rigid-body. After extracting the pose information for the patches, one would need to average the rotations from several patches on a single body segment, to obtain the best estimate of the segment's rotation. Note that depending on the particulars of the post-processing algorithm, the orientation component of the calculated pose may be represented via any of the existing rotational representations; common examples are rotation matrices, quaternions and rotation vectors.

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1.2. Previous Work

Only a few publications exist devoted specifically to the subject of *averaging* rotations [8–10] and we will present the corresponding formulations in detail in Section 2 of this manuscript. In [8], motivated by the sensor-fusion applications in Virtual Reality, two procedures are discussed for averaging rotations: based on the rotation matrices and based on quaternions, the latter advocated for averaging two quaternions only. A more recent reference [9] introduces the problem in the context of robot vision, mentioned earlier, where the orientation of the object is measured with multiple cameras, and also, the problem of registration of medical images. The objective in [9] is to show that the barycentric (or arithmetic) means of rotations, defined based on rotation matrix and quaternion representations, approximate the corresponding means based on the Riemannian metric. The latter is stated as $\phi(\cdot)$ and is the angle of rotation induced by the rotation matrix or the quaternion argument. This metric, further discussed in Section 2, has been used in [11] for computing the mean rotation, in [12] for measuring the distance between two rotation matrices and in [13] to determine the rotation distance between contacting polyhedra. The authors of [10] define two *bi-invariant* metrics for rotation matrices, the notion of bi-invariance to be explained shortly. These are then used as the bases for formulating the corresponding rotation means, referred to as Euclidean and Riemannian, respectively, terms which we use here interchangeably with arithmetic and geometric. The realizations of the two rotation matrix means are derived in [10], specifically, a closed-form solution in case of the Euclidean mean and a set of nonlinear equations to be solved for the Riemannian rotation matrix mean.

A related problem that has received significantly more attention, particularly in the computer graphics and robotics communities, is the problem of *interpolation* of rotations. In this context, one is looking to generate a smooth curve in time, denoted generically as $\mathbf{R}(t)$, which interpolates a specified sequence of rotations at particular time instances. The interpolations can then be used to produce, for example, a smooth motion of a robot end-effector or a camera. In the computer graphics community, use of quanternions for animating rotations has been widely popularized and researched with a number of quaternion-based spline interpolations proposed [14–17]. Other parametrizations of rotation, such as using canonical coordinates [12,17], Cayley-Rodrigues parameters [12] and Euler angles [16] have also been considered for interpolation on the space of orientations. One of the principal issues in rotation interpolation is to develop computationally efficient algorithms [17] which provide sufficiently accurate approximations to the *optimal* interpolation. The latter is typically characterized by the minimum angular acceleration of the resulting curve $\mathbf{R}(t)$, and it produces a smooth interpolated motion.

Central to both the formulation of rotation average and interpolation of rotations is the notion of the underlying *distance measure*, or metric, already mentioned above. This notion must be clearly defined when measuring a distance between two rotations because rotations are not members of a vector space, but belong to $SO(3)$, the special orthogonal group in $\Re^3$. Whichever definition for the metric one proposes, ideally we require that it be *bi-invariant*. This means that if we define a metric between two members of the group of rotations, say $d(\mathbf{R}_1, \mathbf{R}_2)$ denoting the distance between two rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, then it must produce the same measure when evaluated for the pair $(\mathbf{PR}_1\mathbf{Q}, \mathbf{PR}_2\mathbf{Q})$ for every $\mathbf{P}$ and $\mathbf{Q}$ in $SO(3)$. In the context of rotation interpolation, the same bi-invariance property is critical as it requires the orientation curve to be independent of how one selects either the fixed or the moving reference frames [12], when defining the body orientation as a function of time. The notion of appropriate distance metrics on a special *Euclidean* group, $SE(3)$, has been discussed extensively for measuring the distance between *general* rigid-body displacements, particularly in the context of robot trajectory planning [18] and mechanism design [19]. It has been long established that unlike the space of orientations, no bi-invariant metric can be constructed on $SE(3)$ [20], although several propositions for left-invariant [20,21] and frame invariant (objective) solutions [22] have been made.

Having chosen the metric, one can then formulate the corresponding rotation average as the least-squares solution to the corresponding metric-based optimization problem. We note that although bi-invariance is an intuitive and meaningful requirement, it will be shown in this paper that other possible definitions of rotation average are not based on a bi-invariant metric; yet, they can produce excellent estimates of mean rotation. We also suggest that bi-invariance in the sense defined here is possible only for those rotational representations which allow for a multiplicative composition of rotations, or more precisely belong to a multiplicative group.

1.3. About this paper

The present manuscript is organized as follows. In Section 2 we review the existing bi-invariant formulations of the average rotation problem, based primarily on Refs. [8–10], while also referring to their use in literature, and establish clear links between the different solutions for the mean rotation. We then develop a new algebraic realization for the average rotation vector based on the aforementioned Riemannian metric $\phi(\cdot)$. Section 3 is allocated to the presentation and discussion of a non-invariant rotation average, computed as the arithmetic average of rotation vectors. It is included here because averaging of rotation vectors provides a fast and simple to implement solution for the average rotation which, somewhat unexpectedly, is close to the bi-invariant results. It is demonstrated, however, that arithmetic average of rotation vectors is an approximate solution to the Riemannian and Euclidean rotation averages. In Section 4, a summary of the existing and proposed algorithms is presented and their performance evaluated by means of four examples with the average rotations calculated by using solutions from Sections 2 and 3. The examples are comprised of three simulated test-cases and one case where the rotations to be averaged were obtained from pendulum experiments and a marker-based pose measurement system.

2. Existing bi-invariant formulations

2.1. Euclidean formulations

The Euclidean formulation of the rotation average is based on the Euclidean metric for rotation matrices, stated in [10] as:

$$d_F(\mathbf{R}_1,\mathbf{R}_2) = \|\mathbf{R}_1 - \mathbf{R}_2\|_F \tag{1}$$

The above is the Frobenius norm of the difference between two rotation matrices and the same norm is given in [21] and noted to be bi-invariant. Based on this norm, the authors of [10] define the average rotation matrix of a sequence of N rotations $\mathbf{R}_i$ as the solution of the following minimization problem:

$$\bar{\mathbf{R}}_F = arg\ \min_{\mathbf{R}} \sum_{i=1}^{N} \|\mathbf{R}_i - \mathbf{R}\|_F^2 \tag{2}$$

Furthermore, it is demonstrated that the solution for the above average rotation is exactly the orthogonal projection of the arithmetic mean, $\mathbf{R}_{\text{arith}}$, on $SO(3)$. If the arithmetic mean has a positive determinant, this orthogonal projection can be calculated as the unique polar factor of the polar decomposition of $\mathbf{R}_{\text{arith}}$. Hence, one can formulate a simple algorithm to calculate the Euclidean mean rotation matrix in the following two steps:

Algorithm 1. Step 1: Compute $\mathbf{R}_{\text{arith}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{R}_i$

Step 2: Check the determinant of $\mathbf{R}_{\text{arith}}$ and if positive,[1] compute the polar decomposition of $\mathbf{R}_{\text{arith}}$ to get the desired rotation matrix average $\bar{\mathbf{R}}_{\text{F}}$ from [10]:

$$\mathbf{R}_{\text{arith}} = \bar{\mathbf{R}}_F\mathbf{S} \tag{3}$$

where $\mathbf{S}$ is symmetric positive definite, $\mathbf{S} = (\mathbf{R}_{\text{arith}}^{\text{T}}\mathbf{R}_{\text{arith}})^{1/2}$

The formulation in [8] developed earlier than the work in [10] gives a different algorithm for computing the rotation matrix average. In particular, the average of two rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ is defined as:

$$\bar{\mathbf{R}}_{\text{SVD}} = \mathbf{U}\mathbf{V} \tag{4}$$

where $\mathbf{U}\sum V$ is the singular value decomposition (SVD) of the arithmetic sum of the two rotations $\mathbf{R}_1 + \mathbf{R}_2$ and $\sum$ is the diagonal matrix of singular values. The proof of this result, given in [8], demonstrates that this average rotation solves the following minimization problem for the average square penalty:

$$min\int_{|\mathbf{x}|=1}\left[\|(\mathbf{R}_1 - \mathbf{R})\mathbf{x}\|^2 + \|(\mathbf{R}_2 - \mathbf{R})\mathbf{x}\|^2\right]\mathrm{d}\mathbf{x} \tag{5}$$

for displacement between $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}$ rotations 'of the unit sphere' [8][2]. Therefore, the solution for $\mathbf{R}$ above minimizes the effects of the difference between the rotations. The result is directly generalized to N rotations, as well as a weighted average, with the same definition for $\bar{\mathbf{R}}_{SVD}$ as in Eq. (4) and matrices $\mathbf{U}\mathbf{V}$ now from the SVD of the weighed sum, $\sum w_i\mathbf{R}_i$. Exactly the same SVD solution for the average rotation matrix is proposed in [9], where it is also shown that the above SVD-based average solves the following approximate minimization problem:

$$\bar{\mathbf{R}} = arg\ \min_{\mathbf{R}} \sum_{i=1}^{N} \phi^2\left(\mathbf{R}^{-1}\mathbf{R}_i\right) \approx arg\ max\ \mathrm{tr}\left(\mathbf{R}^{-1}\sum_{i=1}^{N}\mathbf{R}_i\right) \tag{6}$$

In Eq. (6), the first equality defines the average rotation based on the Riemannian norm ϕ, while the approximation to arrive at the second statement is based on the second-order Taylor series expansion of the cosine function of ϕ.

One important observation is that the SVD solution of Eq. (4) and the polar decomposition solution in Eq. (3) produce *identical* results for the Euclidean average rotation matrix, which furthermore, as per Eq. (6), represents a second-order approximation of the Riemannian average. Equivalence of the SVD and polar decompositions has been demonstrated in [21] where the two decompositions are employed to realize the embedding of $SE(n-1)$ onto $SO(n)$. The SVD decomposition has also been employed in [23] and [17] to project the interpolant obtained in the ambient matrix space onto the closest member of $SO(3)$. Finally, the same SVD solution was described in Rancourt to produce the maximum likelihood estimator for a sample of rotations 'clustered around their modal value.' The two solutions will be compared in Section 4 of this manuscript.

[1] The determinant check is required to ensure that the result of polar decomposition yields a proper rotation matrix.

[2] Integration in Eq. (5) is carried out over the unit sphere.

In addition to formulating the average rotation matrix, the authors of [9] derive the average quaternion solution for a sequence of N quaternions q_i as:

$$\bar{\mathbf{q}} = \sum_{i=1}^{N} \mathbf{q}_i \Big/ \left\| \sum_{i=1}^{N} \mathbf{q}_i \right\| \tag{7}$$

The above average is *not* based on a bi-invariant norm, but as shown in [9] is an approximate solution of the analogous optimization problem to (6):

$$\bar{\mathbf{q}} = arg\, \min_{\mathbf{q}} \sum_{i=1}^{N} \phi^2\left(\mathbf{q}^* \mathbf{q}_i\right) \tag{8}$$

where the * notation denotes conjugate quaternion and the operation between $\mathbf{q}^*$ and $\mathbf{q}_i$ is standard quaternion multiplication. It is also demonstrated in [9] that the quaternion based solution (7) does not give identical results to the Euclidean rotation matrix solution, and in fact, it provides a more accurate approximation of the Riemannian average.

2.2. Riemannian formulations

Also originally formulated in [10] are the Riemannian bi-invariant metric for rotation matrices and the corresponding rotation mean:

$$d_R(\mathbf{R}_1, \mathbf{R}_2) = \frac{1}{\sqrt{2}} \| log\left(\mathbf{R}_1^T \mathbf{R}_2\right) \|_F \tag{9}$$

$$\bar{\mathbf{R}}_R = arg\, \min_{\mathbf{R}} \sum_{i=1}^{N} \| log\left(\mathbf{R}_1^T \mathbf{R}_2\right) \|_F^2 \tag{10}$$

and as one can see, these make use of the principal matrix logarithm. It is noted in [10] that the distance measure Eq. (9) represents the arc-length of the shortest geodesic curve and it lies entirely in $SO(3)$, which is not the case for its Euclidean counterpart Eq. (1). The characterization of the above mean is more complicated than in the Euclidean case as solution of the corresponding minimization problem Eq. (10) is more involved. It is shown in [10] that a necessary but not sufficient condition for the minima is:

$$\sum_{i=1}^{N} log\left(\mathbf{R}_i^T \mathbf{R}\right) = 0 \tag{11}$$

which gives a matrix nonlinear equation for the average rotation matrix $\mathbf{R}$. The evaluation of Eq. (11) is of course subject to the limitations of the principal logarithm of a matrix: it exists for positive semi-definite matrices only. An alternative formulation of the Riemannian metric, already alluded to in Section 2.1 is:

$$d_{R,\phi}(\mathbf{R}_1, \mathbf{R}_2) = \phi\left(\mathbf{R}_1^T \mathbf{R}_2\right) \tag{12}$$

and it is the angle induced by the relative rotation between $\mathbf{R}_1$ and $\mathbf{R}_2$. The exact same measure can be computed using quaternions, that is:

$$d_{q,\phi}(\mathbf{q}_1, \mathbf{q}_2) = \phi\left(\mathbf{q}_1^* \mathbf{q}_2\right) = \phi\left(\mathbf{R}_1^T \mathbf{R}_2\right) \tag{13}$$

Note that the order of rotations used in Eqs. (12) and (13) is inconsequential. It can be shown, using the properties of the principal logarithm for a matrix in $SO(3)$ that the two rotation-matrix based expressions for the Riemannianmetric, i.e., Eqs. (9) and (12) produce *identical* measures [10,12]. The average rotation matrix corresponding to the angle metric of Eq. (12) was already defined in Eq. (6) but is restated here as:

$$\bar{\mathbf{R}}_R = arg\, \min_{\mathbf{R}} \sum_{i=1}^{N} (\Delta\phi_i)^2 \tag{14}$$

where we make a slight change of notation to $\Delta\phi_i = \phi(\mathbf{R}_i^T \mathbf{R})$. The corresponding formulation in terms of quaternions is given in Eq. (8).

Formulating the Riemannian mean as a solution to the angle-based optimization problem Eq. (14) allows us to derive a realization different from the one in Eq. (11), presented here for the first time. In particular, letting ϕ_R denote the average vector associated with the average rotation $\bar{\mathbf{R}}_R$, one can show that the gradient of the objective function $f = f(\phi) = \sum_{i=1}^{N} (\Delta\phi_i)^2$ is:

$$\frac{\partial f}{\partial \boldsymbol{\phi}} = \sum_{i=1}^{N} 2\Delta\phi_i \frac{\partial \Delta\phi_i}{\partial \boldsymbol{\phi}} \tag{15}$$

where

$$\frac{\partial \Delta\phi_i}{\partial \boldsymbol{\phi}} = -\frac{1}{2\sin\Delta\phi_i}\mathbf{g}_i$$

and

$$\mathbf{g}_i = \frac{2}{\phi}[(1-\cos\phi_i)(1-\cos\phi)(\mathbf{f}_i\cdot\mathbf{f}) + \sin\phi_i\sin\phi]\mathbf{f}_i + \left[\left(\sin\phi - \frac{2(1-\cos\phi)}{\phi}\right)(1-\cos\phi_i)(\mathbf{f}_i\cdot\mathbf{f})^2 - (1+\cos\phi_i)\sin\phi + 2\sin\phi_i\left(\cos\phi - \frac{\sin\phi}{\phi}\right)(\mathbf{f}_i\cdot\mathbf{f})\right]\mathbf{f} \tag{16}$$

Detailed derivation of the result above is included in Appendix A and it makes use of the tensorial representation of $\mathbf{R}$ in terms of the rotation vector ϕ:

$$\mathbf{R} = (1-\cos\phi)\mathbf{f}\otimes\mathbf{f} + \cos\phi\mathbf{I} - \sin\phi\epsilon\mathbf{f} \tag{17}$$

In the above, $\phi = |\boldsymbol{\phi}|$, $\mathbf{f} = \frac{\boldsymbol{\phi}}{\phi}$ is the Euler axis of rotation, $\otimes$ denotes a tensor product, $\cdot$ denotes the dot product (generalized to tensors) and ϵ is the permutation tensor [24]. Setting the gradient to zero provides a necessary condition for the minima and this gives a set of three nonlinear equations for the Riemannian average rotation vector $\boldsymbol{\phi}_R$:

$$\sum_{i=1}^{N} \frac{\Delta\phi_i}{\sin\Delta\phi_i}\mathbf{g}_i = 0 \tag{18}$$

3. Averaging of rotation vectors

It is tempting and somewhat intuitive to use the arithmetic average of rotation vectors to define an average rotation, that is:

$$\bar{\boldsymbol{\phi}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\phi}_i \tag{19}$$

The above is analogous to the arithmetic quaternion mean (normalized) stated in Eq. (7) of this paper, and likewise, its underlying metric is not bi-invariant. In particular, the distance measure between two rotation vectors is the standard Euclidean norm:

$$d(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|_2 = \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\| \tag{20}$$

To establish that this norm is *not* bi-invariant, we need to form the corresponding rotation matrices, $\mathbf{R}_1(\boldsymbol{\phi}_1)$ and $\mathbf{R}_2(\boldsymbol{\phi}_2)$, using Eq. (18) for example or its matrix counterpart iteParkandKang, and then the transformed matrices $\hat{\mathbf{R}}_1 = \mathbf{P}\mathbf{R}_1\mathbf{Q}$ and $\hat{\mathbf{R}}_2 = \mathbf{P}\mathbf{R}_2\mathbf{Q}$ for arbitrary $\mathbf{P}$ and $\mathbf{Q}$ in $SO(3)$. From the transformed rotation matrices, we determine the corresponding rotation vectors as $\hat{\boldsymbol{\phi}}_1 = \hat{\boldsymbol{\phi}}_1\left(\hat{\mathbf{R}}_1\right)$ and $\hat{\boldsymbol{\phi}}_2 = \hat{\boldsymbol{\phi}}_2\left(\hat{\mathbf{R}}_2\right)$ and check for bi-invariance by comparing the two distance values $d(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ and $d\left(\hat{\boldsymbol{\phi}}_1, \hat{\boldsymbol{\phi}}_2\right)$. Any non-trivial example (i.e., two original rotations are not about the same axes) will expose the lack of invariance of the metric (21).

We now demonstrate that the rotation vector mean in Eq. (19) is a *first-order* approximation of the Riemannian mean computed from Eq. (18). To this end, let us define the difference vectors $\boldsymbol{\delta}_i$ between individual rotations $\boldsymbol{\phi}_i$ and the Riemannian average $\boldsymbol{\phi}_R$ as:

$$\boldsymbol{\phi}_i = \boldsymbol{\phi}_R + \boldsymbol{\delta}_i \tag{21}$$

It then follows that the arithmetic average of Eq. (20) can be expressed as

$$\bar{\boldsymbol{\phi}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\phi}_i = \boldsymbol{\phi}_R + \bar{\boldsymbol{\delta}} \tag{22}$$

where we introduced the average difference vector:

$$\bar{\boldsymbol{\delta}} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\delta}_i \tag{23}$$

Then, assuming that ϕ (the norm of $\boldsymbol{\phi}_R$) is non-zero, to first order in δ_i we obtain:

$$\phi_i = \phi + \mathbf{f}\cdot\boldsymbol{\delta}_i, \qquad \mathbf{f}_i = (1-\mathbf{f}\cdot\boldsymbol{\delta}_i/\phi)\mathbf{f} + \boldsymbol{\delta}_i/\phi, \qquad \mathbf{f}_i\cdot\mathbf{f} = 1 \tag{24}$$

Substituting from the above into Eq. (17), expanding and simplifying, it is possible to show that to first order in δ_i the expression for $\mathbf{g}_i$ reduces to:

$$\mathbf{g}_i = 2\left(1-\frac{2(1-\cos\phi)}{\phi^2}\right)(\mathbf{f}\cdot\boldsymbol{\delta}_i)\mathbf{f} + \frac{4(1-\cos\phi)}{\phi^2}\boldsymbol{\delta}_i \tag{25}$$

and furthermore,

$$\frac{\Delta\phi_i}{\sin\Delta\phi_i} = 1 \tag{26}$$

Accordingly, the solution of Eq. (19) to first order requires that:

$$\left(1-\frac{2(1-\cos\phi)}{\phi^2}\right)\left(\mathbf{f}\cdot\bar{\boldsymbol{\delta}}\right)\mathbf{f} + \frac{2(1-\cos\phi)}{\phi^2}\bar{\boldsymbol{\delta}} = 0 \tag{27}$$

Finally, taking the dot product of Eq. (28) with $\mathbf{f}$ yields $\mathbf{f}\cdot\bar{\boldsymbol{\delta}} = 0$ which for nonzero ϕ requires that:

$$\bar{\boldsymbol{\delta}} = 0 \tag{28}$$

The above result, valid for *any* nonzero $\boldsymbol{\phi}_R$, states that to first order deviations of rotations in the sequence from their average, the Riemannian average rotation vector is equal to the simple arithmetic average of rotation vectors as given in Eq. (19). In Appendix B, we also prove that the arithmetic average of rotation vectors provides an approximate solution for the Euclidean mean of Section 2.1, which is second-order in both the average and sample rotation angles. As shown in Section 4, this average produces excellent estimates for the Riemannian and Euclidean means for many practical situations, and still reasonable estimates, even when rotations are large and very different from each other.

4. Numerical results

4.1. Summary of algorithms

Before proceeding to numerical results, let us briefly summarize the algorithms. As noted in Section 2, the polar decomposition and the SVD solutions for the average rotation matrix yield *identical* results, assuming the polar decomposition exists. Based on our discussion in Section 2, it also follows that this rotation matrix average is the exact solution to the minimization problem (2), which recall is based on the bi-invariant Euclidean metric. As well, the Euclidean average rotation represents an *approximate* solution to optimization problem (14), the latter based on the bi-invariant Riemannian metric.

Solution of the nonlinear Eq. (18) with Eq. (16) yields the average rotation vector, with the corresponding rotation matrix calculated if desired, and it is the exact solution to the Riemannian optimization problem (14). This average is identical to the average obtained from the principal logarithm realization (11), when the latter exists. It is also noted that for the case of *two* rotations only, the Euclidean and Riemannian averages are always identical [10].

We discussed two solutions that are not based on a bi-invariant distance measure. The quaternion solution (7), similarly to the Euclidean average rotation matrix, also gives a second-order approximation to the Riemannian average, but it is apparently more accurate [9]. Lastly we have a solution described in Section 3, not based on a bi-invariant metric: the arithmetic average of rotation vectors. It was demonstrated in Section 3 that direct averaging of rotation vectors represents an approximate solution to the Riemannian and Euclidean means.

Accordingly, the following methods have been implemented for calculating the average rotation:

Euclidean average
- Eu-SVD: decomposition of the arithmetic mean of rotation matrices, Eq. (4);
- Eu-PD: the polar decomposition of the arithmetic mean of rotation matrices, Eq. (3).

Riemannian average
- Ri-M: solution of nonlinear Eq. (11) for average rotation *matrix*;
- Ri-RV: solution of nonlinear Eq. (19) for average rotation *vector*.

Table 1
Rotation angle computed with averaging methods for test case 1.

Averaging method	Average angle (rad)
Eu-SVD, Eu-PD, Ri-M, Ri-RV, QAA	$\bar{\theta}_{calc} = 0.4084$
RVA	$\bar{\theta}_{calc} = 0.3927$

Table 2
Rotation angle and axis computed with averaging methods for test case 2 (second column) and test case 3 (third column).

Method	Average angle (rad) and axis	
	Test case 2	Test case 3
Eu-SVD, Eu-PD	$\bar{\phi}_{calc} = 0.6165$ $\bar{\mathbf{f}}_{calc} = [0.5476, 0.6006, 0.5826]$	$\bar{\phi}_{calc} = 1.0065$, $\bar{\mathbf{f}}_{calc} = [0.5774, 0.8165, 0]$
Ri-M	$\bar{\phi}_{calc} = 0.6161$ $\bar{\mathbf{f}}_{calc} = [0.5472, 0.6005, 0.5831]$	Solver fails
Ri-RV	$\bar{\phi}_{calc} = 0.6161$ $\bar{\mathbf{f}}_{calc} = [0.5472, 0.6005, 0.5831]$	$\bar{\phi}_{calc} = 1.4556$, $\bar{\mathbf{f}}_{calc} = [0.3567, 0.5045, 0.7863]$
QAA	$\bar{\phi}_{calc} = 0.6162$ $\bar{\mathbf{f}}_{calc} = [0.5473, 0.6006, 0.5830]$	$\bar{\phi}_{calc} = 1.3984$, $\bar{\mathbf{f}}_{calc} = [0.3789, 0.5345, 0.7559]$
RVA	$\bar{\phi}_{calc} = 0.6123$ $\bar{\mathbf{f}}_{calc} = [0.5469, 0.6007, 0.5831]$	$\bar{\phi}_{calc} = 1.2217$, $\bar{\mathbf{f}}_{calc} = [0.2857, 0.4286, 0.8571]$

Quaternion average
  QAA: normalized arithmetic mean of quaternions as per Eq. (7).
Rotation vector average
  RVA: arithmetic mean of rotation vectors as per Eq. (20).

In the following, we present results obtained using the MATLAB implementation[3] of the above algorithms, for four test cases. The results will be compared by using the angle-axis representations of the corresponding averages. It is also noted that the implementation and basic validity of all methods discussed in this paper have been verified with a baseline test comprised of a sequence of rotations about a fixed arbitrary axis, and different rotation angles in the specified range. Here, all methods predict the expected average: a rotation about the same fixed axis through an angle which is the arithmetic average of the rotation angles of the sequence.

4.2. Test case 1

This test case was taken from Ref. [10] and it has analytical geometric and arithmetic average solutions, which as shown in Ref. [10] coincide. We consider a sequence of N rotations, each through a fixed specified angle θ about the axis defined by one of the unit vectors $\mathbf{n}_i = [\sin\alpha\cos\beta_i, \sin\alpha\sin\beta_i, \cos\alpha]^T$, where $\beta_i = \frac{2(i-1)\pi}{N}, i = 1, \ldots, N$ and the angle α has a specified fixed value. The special case for $N=2$ can be used to describe two rotations about any two axes, through the same angle of rotation. Since the rotation axes are symmetric about the z-axis, we expect the average rotation to be about that axis. The analytical solution for the average angle $\bar{\theta}$ is derived in [10] as:

$$tan\frac{\bar{\theta}}{2} = cos\,\alpha\, tan\frac{\theta}{2} \tag{29}$$

Results from all methods are presented in Table 1 for the following parameter settings: $N=50, \theta=\pi/4, \alpha=\pi/3$. In this case as expected, the Euclidean and Riemannian solutions are identical between each other and reproduce the analytical solution of Eq. (29), which is also reproduced by the quaternion arithmetic average. The rotation vector averaging generates the correct rotation axis, the z-axis, but predicts a slightly different average angle.

4.3. Test case 2

The rotation matrices for our second test case were generated as per the statistical model in Ref. [1] of a sample of N rotations, clustered around their mean value $\bar{\mathbf{R}}$:

$$\mathbf{R}_i = \bar{\mathbf{R}}\, exp\, \boldsymbol{\phi}_i^{\times}, \quad i = 1 \ldots N$$

[3] For all algorithms, standard MATLAB functions were used where possible. For example, the logarithm of a matrix was computed using the 'logm' function.

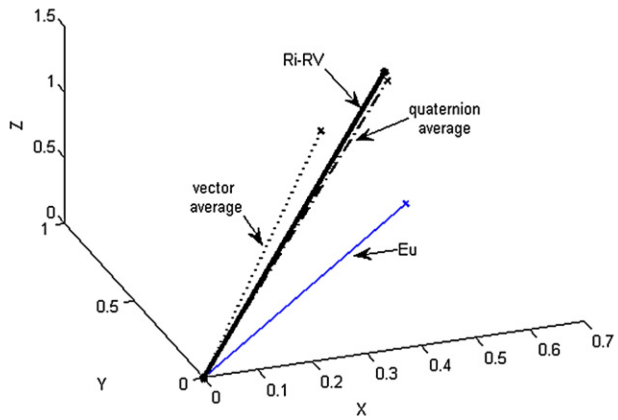

Fig. 1. Average rotation vectors for test case 3 from arithmetic, geometric and rotation vector averaging solutions.

where $\boldsymbol{\phi}_i$ are normally distributed about the origin with standard deviation σ and whose components are small. Numerical results displayed in Table 2 were obtained with $N=100$, $\sigma=0.2$ and an arbitrarily chosen mean rotation, corresponding to the mean rotation vector $\bar{\boldsymbol{\phi}}=\frac{\pi}{5}\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]^T$. As observed from Table 2[4] (middle column) and as one would expect since all rotations in the sequence are close to each other, Riemannian and Euclidean solutions are nearly identical (differences appear in fourth decimal place) and are also close to the arithmetic rotation vector average. In line with theoretical predictions, the quaternion average provides the closest approximation of the Riemannian mean.

4.4. Test case 3

Different from the previous two examples, this test case was chosen to accentuate the differences between the various rotation means. Thus, we consider averaging of three principal rotations about x- , y- and z-axis, through different and large angles: $\theta_x=\pi/3$, $\theta_y=\pi/2$, $\theta_z=\pi$. As shown in Table 2 (third column), the arithmetic and geometric averages are significantly different: the arithmetic solution produces the average rotation axis which lies in the x–y plane, essentially 'discounting' the rotation through π about the z-axis; it is clearly unacceptable for this test case. We also note that the matrix logarithm solution of Eq. (11) fails to produce the correct average because the principal matrix logarithm cannot be obtained for some of the terms. The geometric and quaternion averages are very close, while the rotation vector average, although somewhat different, produces reasonable results. The average rotation vectors corresponding to the four solutions are illustrated graphically in Fig. 1, where the differences in the solutions are clearly visible.

4.5. Test case 4

The data for this last test case was generated with a pendulum experimental set-up, a photograph shown in Fig. 2. The pendulum is comprised of a rod to which a rigid mass is attached via two springs, thus allowing the mass to oscillate along the pendulum while the rod rotates about the hinge. Five reflective markers were affixed to the mass and their positions measured in the laboratory frame using the Vicon infrared camera system. The 3D positions of the markers were processed to generate rotation matrices of the mass-fixed frame as a function of time, at the rate of 500 Hz. The resulting angle of the pendulum as a function of time is plotted in Fig. 3 (right) and the rotation vectors ϕ_i are shown in Fig. 3 (left).

Our goal in the present test case is to determine the best estimate only for the rotation *axis* of the pendulum in the laboratory frame. Because the data set is quite large (2090 rotations to be averaged), we will also use this example to compare the computational performance of the different methods. As per results of the previous test cases, we employ the arithmetic average of rotation vectors (RVA) as an initial guess for the solution of nonlinear equations for the Ri-RV and Ri-M averages; we found that this provides nearly a factor of two speed-up compared to starting the solver with an arbitrary initial guess. Note that only the data corresponding to the positive rotation angle in Fig. 3, i.e., the right half of the point cloud, is employed for the sample to be averaged, since due to symmetry of rotation vectors, using the complete data set leaves primarily numerical noise for averaging.

The rotation *axes* predicted with the arithmetic, geometric and quaternion averages are identical to five decimal places or better, specifically, $\bar{\mathbf{f}}_{\text{calc}}=[0.06137,-0.99338,-0.09713]^T$, while the result computed by averaging rotation vectors differs in the fourth decimal, $\bar{\mathbf{f}}_{\text{calc}}=[0.06138,-0.99338,-0.09714]^T$. Graphical illustration of the computed rotation axes is shown in Fig. 4 where we overlay the four solutions on the original data set of rotation vectors and extend the rotation axes beyond the point cloud in order to make them visible. As one can see, the four rotation axes computed with the averaging algorithms are

[4] Since the sample is generated with a random distribution of $\bar{\phi}$, results in Table 2 are from a particular run, but are representative.

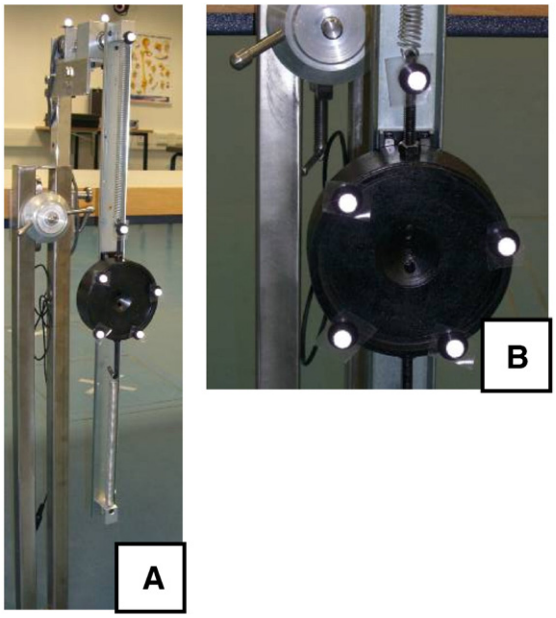

Fig. 2. Pendulum experimental set-up for test case 4.

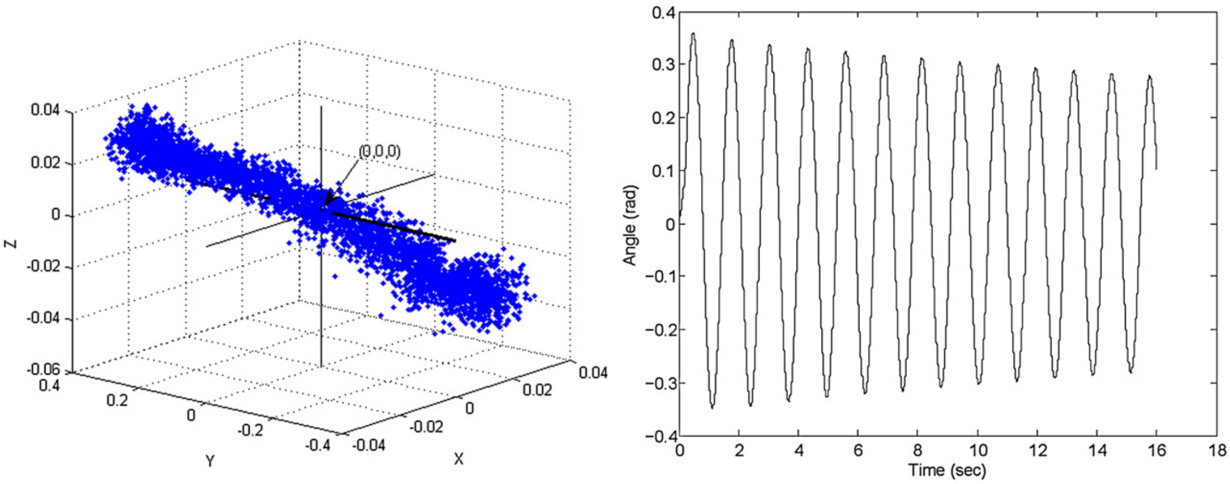

Fig. 3. Endpoints of 4000 rotation vectors (left) and pendulum angle (right) for test case 4.

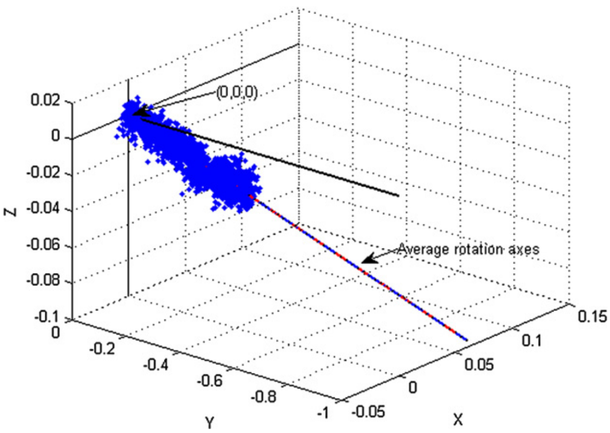

Fig. 4. Predicted rotation axes (arithmetic, geometric and rotation vector averages) for test case 4 overlayed on rotation vector data.

Table 3
Computational performance of averaging methods for test case 4.

Method	Eu-SVD	Eu-PD	Ri-M	Ri-RV	QAA	RVA
CPU time (s)	0.03	0.06	48.8	4.4	0.0	0.0

indistinguishable on this plot. Comparison of the computational times presented in Table 3 shows that among the methods considered, simple averaging of rotation vectors and quaternions requires negligible CPU time[5] and it is followed closely by the two Euclidean methods. The new realization of the Riemannian average presented in this paper is substantially faster than the log-based Riemannian average, which takes by far the longest time to compute.

5. Conclusions

This paper presents an expose of the various methods for calculating the average of rigid-body rotations. The solutions can be categorized according to whether the underlying metric is bi-invariant or not, and those in the bi-invariant category, as Euclidean or Riemannian. Among the two possible algorithms for calculating the Euclidean rotation matrix average, the SVD solution is preferable because it is more general and computationally robust. In the Riemannian category, we suggest that the solution based on the angle measure, with the new realization developed in this paper, is more robust than the characterization involving the principal matrix logarithm, in addition to being substantially faster. Our numerical examples show that under rather 'extreme' conditions of large-angle rotations about orthogonal axes, the differences between the Euclidean and Riemannian averages can be very significant. On the other hand, when averaging rotations that are expected to be close, as is likely to be the case in most applications, all solutions predict nearly identical results, including the simple arithmetic averages of quaternions and rotation vectors. Indeed, the latter is probably the most robust technique as it is not subject to the fickle nature of nonlinear equations or optimization solvers, nor does it require a division (normalization) operation as is the case for quaternion averaging. Both quaternion and rotation vector averaging are computationally trivial and therefore, are good candidates for real-time applications, where speed is of paramount importance. If one absolutely desires a Riemannian average, and computational and implementation considerations are not important, the novel realization presented in this paper for computing the Riemannian average is recommended.

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Appendix A. Detailed derivation of Eq. (16).

We derive here the result:

$$\frac{\partial \Delta\phi_i}{\partial \boldsymbol{\phi}} = -\frac{1}{2\,sin\,\Delta\phi_i}\mathbf{g}_i \tag{A – 1}$$

where $\mathbf{g}_i$ is given by Eq. (16). Throughout the derivation, we will make use of the dot product between two second-order tensors defined as:

$$\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}\left(\mathbf{A}\mathbf{B}^T\right) \tag{A – 2}$$

We begin with the defining relations for the angle $\Delta\phi_i$ and rotation matrices $\mathbf{R}_i$ and $\mathbf{R}$ as:

$$cos\,\Delta\phi_i = \frac{1}{2}(\Delta\mathbf{R}_i\cdot\mathbf{I}-1), \quad 0\leq\Delta\phi_i\leq\pi \tag{A – 3}$$

where

$$\Delta\mathbf{R}_i = \mathbf{R}_i^T\mathbf{R}$$
$$\mathbf{R}(\boldsymbol{\phi}) = (1-cos\,\phi)\mathbf{f}\otimes\mathbf{f} + cos\,\phi\mathbf{I}-sin\,\phi\epsilon\mathbf{f}, \quad \phi = |\boldsymbol{\phi}|, \quad \mathbf{f} = \frac{\boldsymbol{\phi}}{\phi}$$

[5] The CPU times quoted are for the algorithm proper, i.e., they do not include the conversion time that may be required to obtain the pool of rotation vectors or quaternions.

and

$$\mathbf{R}_i(\phi_i) = (1-cos\,\phi_i)\mathbf{f}_i\otimes\mathbf{f}_i + cos\,\phi_i\mathbf{I}-sin\,\phi_i\epsilon\mathbf{f}_i,\quad \phi_i = |\boldsymbol{\phi}_\mathrm{i}|,\quad \mathbf{f}_i = \frac{\boldsymbol{\phi}_i}{\phi_i}$$

Using the following tensor identities [24]:

$$(\epsilon\mathbf{f}_i)\cdot(\mathbf{f}\otimes\mathbf{f}) = 0,\quad (\epsilon\mathbf{f}_i)\cdot\mathbf{I} = 0,\quad (\epsilon\mathbf{f}_i)\cdot(\epsilon\mathbf{f}) = 2\mathbf{f}_i\cdot\mathbf{f},$$

it can be shown that:

$$\Delta\mathbf{R}_i\cdot\mathbf{I} = \mathbf{R}_i\cdot\mathbf{R} = (1-cos\phi_i)(1-cos\phi)(\mathbf{f}_i\cdot\mathbf{f})^2 + cos\phi_i + (1 + cos\phi_i)cos\phi + 2sin\phi_i sin\phi(\mathbf{f}_i\cdot\mathbf{f}) \tag{A – 4}$$

Substituting from Eq. (A-4) into Eq. (A-3) and differentiating the result with respect to time we obtain:

$$\begin{aligned}-2sin\Delta\phi_i\frac{\mathrm{d}\Delta\phi_i}{\mathrm{d}t} = \frac{\mathrm{d}(\Delta\mathbf{R}_i\cdot\mathbf{I})}{\mathrm{d}t} &= (1-cos\,\phi_i)(sin\,\phi)\dot{\phi}(\mathbf{f}_i\cdot\mathbf{f})^2 + 2(1-cos\,\phi_i)(1-cos\,\phi)(\mathbf{f}_i\cdot\mathbf{f})(\mathbf{f}_i\cdot\dot{\mathbf{f}})-(1 + cos\,\phi_i)(sin\,\phi)\dot{\phi}\\ &+ 2(sin\,\phi_i\,cos\,\phi)\dot{\phi}(\mathbf{f}_i\cdot\mathbf{f}) + 2\,sin\,\phi_i\,sin\,\phi(\mathbf{f}_i\cdot\mathbf{f})\end{aligned} \tag{A – 5}$$

Now collecting the $\mathbf{f}_i\cdot\dot{\mathbf{f}}$ and $\dot{\phi}$ terms yields:

$$\begin{aligned}\frac{\mathrm{d}\Delta\phi_i}{\mathrm{d}t} = &-\frac{1}{2\,sin\,\Delta\phi_i}\Big[[2(1-cos\,\phi_i)(1-cos\,\phi)(\mathbf{f}_i\cdot\mathbf{f}) + 2\,sin\,\phi_i\,sin\,\phi]\left(\mathbf{f}_i\cdot\dot{\mathbf{f}}\right)\\ &+ \left[(1-cos\,\phi_i)(sin\,\phi)(\mathbf{f}_i\cdot\mathbf{f})^2-(1 + cos\,\phi_i)(sin\,\phi) + 2(sin\,\phi_i\,cos\,\phi)(\mathbf{f}_i\cdot\mathbf{f})\right]\dot{\phi}\Big]\end{aligned} \tag{A – 6}$$

At this point, we invoke the following expressions for the time derivatives $\dot{\phi}$ and $\dot{\mathrm{f}}$:

$$\dot{\phi} = \mathbf{f}\cdot\dot{\boldsymbol{\phi}},\quad \mathbf{f} = \frac{1}{\phi}\left(\dot{\boldsymbol{\phi}}-\left(\mathbf{f}\cdot\dot{\boldsymbol{\phi}}\right)\mathbf{f}\right) \tag{A – 7}$$

Substituting from the above into Eq. (A-6) and rearranging yields:

$$\begin{aligned}\frac{\mathrm{d}\Delta\phi_i}{\mathrm{d}t} = &-\frac{1}{2\,sin\,\Delta\phi_i}\left[\left(\left(sin\phi-\frac{2(1-cos\,\phi)}{\phi}\right)(1-cos\,\phi_i)(\mathbf{f}_i\cdot\mathbf{f})^2 + 2\,sin\,\phi_i\left(cos\phi-\frac{sin\,\phi}{\phi}\right)(\mathbf{f}_i\cdot\mathbf{f})-(1 + cos\,\phi_i)sin\,\phi\right)\left(\mathbf{f}\cdot\dot{\boldsymbol{\phi}}\right)\right.\\ &\left.+ \frac{2}{\phi}((1-cos\phi_i)(1-cos\phi)(\mathbf{f}_i\cdot\mathbf{f}) + sin\phi_i sin\phi)\left(\mathbf{f}_i\cdot\dot{\boldsymbol{\phi}}\right)\right]\end{aligned} \tag{A – 8}$$

Finally, the above can be rewritten as:

$$\frac{\mathrm{d}\Delta\phi_i}{\mathrm{d}t} = -\frac{1}{2sin\Delta\phi_i}\mathbf{g}_i\cdot\dot{\boldsymbol{\phi}} \tag{A – 9}$$

with $\mathbf{g}_i$ as given in Eq. (17). To complete the derivation, we observe that:

$$\frac{\mathrm{d}\Delta\phi_i}{\mathrm{d}t} = \frac{\partial\Delta\phi_i}{\partial\boldsymbol{\phi}}\cdot\dot{\boldsymbol{\phi}} \tag{A – 10}$$

and thus comparing Eqs. (A-10) and (A-9) leads immediately to the desired result (A-1).

Appendix B. Arithmetic average of rotation vectors is a second-order approximation of the Euclidean mean-proof

We demonstrate that the rotation vector mean in Eq. (19) is an approximation of the Euclidean mean which is itself an approximation of the Riemannian mean as stated in Eq. (6). In particular, starting with the right-hand side of Eq. (6), which was derived in [9] by making use of the second-order Taylor expansion of the cosine function and using the notation adopted in Eq. (14), we have:

$$cos\Delta\phi_i = \frac{\mathrm{tr}\left(\mathbf{R}_i^T\mathbf{R}\right)-1}{2}\approx 1-\frac{1}{2}\Delta\phi_i^2 \tag{B – 1}$$

and hence Eq. (6) in our notation takes the form:

$$arg\ \underset{\mathbf{R}}{min} \sum_{i=1}^{N} (\Delta\phi_i)^2 \approx arg\ max \sum_{i=1}^{N} \mathrm{tr}\left(\mathbf{R}_i^T \mathbf{R}\right) \tag{B-2}$$

where the right-hand side is simply an alternative statement of the Euclidean mean of Eq. (2). We now explicitly evaluate the trace of $\mathbf{R}_i^T\mathbf{R}$ on the right-hand side of Eq. (B-2) using the expression for the rotation tensor as a function of the rotation vector, given earlier in Eq. (18). Again, by employing the second-order approximation of the cosine and sine functions, and substituting for $\mathbf{f} = \frac{\boldsymbol{\phi}}{\phi}$ Eq. (18) simplifies to:

$$\mathbf{R} = \frac{1}{2}\boldsymbol{\phi}\otimes\boldsymbol{\phi} + \left(1-\frac{1}{2}\phi^2\right)\mathbf{I}-\epsilon\boldsymbol{\phi} \tag{B-3}$$

and similarly for $\mathbf{R}_i^T$:

$$\mathbf{R}_i^T = \frac{1}{2}\boldsymbol{\phi}_i\otimes\boldsymbol{\phi}_i + \left(1-\frac{1}{2}\phi_i^2\right)\mathbf{I} + \epsilon\boldsymbol{\phi}_i \tag{B-4}$$

Evaluating the product $\mathbf{R}_i^T\mathbf{R}$ with the above and expanding the trace operation symbolically, we obtain:

$$\mathrm{tr}\left(\mathbf{R}_i^T\mathbf{R}\right) = 3-\left(\phi^2 + \phi_i^2-2\boldsymbol{\phi}\cdot\boldsymbol{\phi}_i\right) + \frac{1}{4}\left((\boldsymbol{\phi}\cdot\boldsymbol{\phi}_i)^2 + \phi^2\phi_i^2\right) \tag{B-5}$$

which to second order reduces to:

$$\mathrm{tr}\left(\mathbf{R}_i^T\mathbf{R}\right)\approx 3-\left(\phi^2 + \phi_i^2-2\boldsymbol{\phi}\cdot\boldsymbol{\phi}_i\right) = 3-\|\boldsymbol{\phi}_i-\boldsymbol{\phi}\|^2 \tag{B-6}$$

Finally, substituting this result into the right-hand side of Eq. (B-2), we obtain the approximation of the Euclidean mean as the least-squares formulation of the average rotation vector:

$$arg\ max \sum_{i=1}^{N} \mathrm{tr}\left(\mathbf{R}_i^T\mathbf{R}\right)\approx arg\ \underset{\boldsymbol{\phi}}{min} \sum_{i=1}^{N} \|\boldsymbol{\phi}_i-\boldsymbol{\phi}\|^2 \tag{B-7}$$

Solution of the second optimization problem above is exactly the arithmetic average rotation vector of Eq. (19).

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